Exercise 5.2.7

Solve these equations either algebraically or graphically.

1 a $2x + y = 7$, $3x + 2y = 12$ b $5x + 4y = 21$, $x + 2y = 9$ c $x + y = 7$, $3x + 4y = 23$

d $2x - 3y = -3$, $3x + 2y = 15$ e $4x = 4y + 8$, $x + 3y = 10$ f $x + 5y = 11$, $2x - 2y = 10$

2 a $x + y = 5$, $3x - 2y + 5 = 0$ b $2x - 2y = 6$, $x - 5y = -5$ c $2x + 3y = 15$, $2y = 15 - 3x$

d $x - 6y = 0$, $3x - 3y = 15$ e $2x - 5y = -11$, $3x + 4y = 18$ f $x + y = 5$, $2x - 2y = -2$

3 a $3y = 9 + 2x$, $3x + 2y = 6$ b $x + 4y = 13$, $3x - 3y = 9$ c $2x = 3y - 19$, $3x + 2y = 17$

d $2x - 5y = -8$, $-3x - 2y = -26$ e $5x - 2y = 0$, $2x + 5y = 29$ f $8y = 3 - x$, $3x - 2y = 9$

4 a $4x + 2y = 5$, $3x + 6y = 6$ b $4x + y = 14$, $6x - 3y = 3$ c $10x - y = -2$, $-15x + 3y = 9$

d $-2y = 0.5 - 2x$, $6x + 3y = 6$ e $x + 3y = 6$, $2x - 9y = 7$ f $5x - 3y = -0.5$, $3x + 2y = 3.5$

5.3 Right-angled trigonometry

Lord Nelson would have used trigonometry in navigation

Trigonometry and the trigonometric ratios developed from the ancient study of the stars. The study of right-angled triangles probably originated with the Egyptians and the Babylonians, who used them extensively in construction and engineering.

The ratios, which are introduced in this chapter, were set out by Hipparchus of Rhodes about 150 BC.

Trigonometry was used extensively in navigation at sea, particularly in the sailing ships of the eighteenth and nineteenth centuries, when it formed a major part of the examination to become a lieutenant in the Royal Navy.

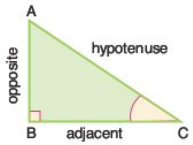
The trigonometric ratios

There are three basic trigonometric ratios: sine, cosine and tangent and you should already be familiar with these.

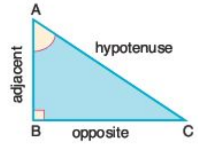
Each of the trigonometric ratios relates an angle of a right-angled triangle to a ratio of the lengths of two of its sides.

The sides of the triangle have names, two of which are dependent on their position in relation to a specific angle.

The longest side (always opposite the right angle) is called the **hypotenuse**. The side opposite the angle is called the **opposite** side and the side next to the angle is called the **adjacent** side.



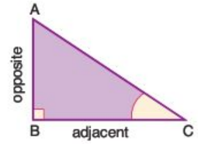
Note that, when the chosen angle is at A, the sides labelled opposite and adjacent change.



Tangent

The tangent ratio is:

$$\tan C = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$$



Worked examples

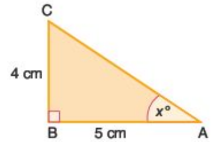
1 Calculate the size of angle BAC.

$\tan x° = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{5}$

$x = \tan^{-1}\left(\frac{4}{5}\right)$

$x = 38.7$ (3 s.f.)

$\angle BAC = 38.7°$ (3 s.f.)



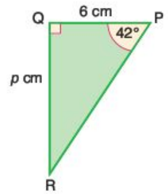
2 Calculate the length of QR.

$\tan 42° = \frac{p}{6}$

$6 \times \tan 42° = p$

$p = 5.40$ (3 s.f.)

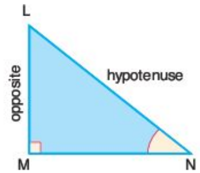
QR = 5.40 cm (3 s.f.)



Sine

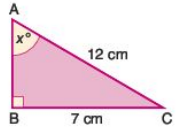
The sine ratio is:

$$\sin N = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$$



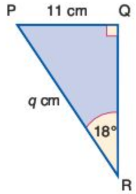
Worked examples

1 Calculate the size of angle BAC.



$$\sin x = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{7}{12}$$
$$x = \sin^{-1}\left(\frac{7}{12}\right)$$
$$x = 35.7 \text{ (3 s.f)}$$
$$\angle BAC = 35.7° \text{ (3 s.f)}$$

2 Calculate the length of PR.

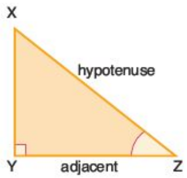


$$\sin 18° = \frac{11}{q}$$
$$q \times \sin 18° = 11$$
$$q = \frac{11}{\sin 18°}$$
$$q = 35.6 \text{ (3 s.f.)}$$
$$PR = 35.6\,\text{cm (3 s.f.)}$$

Cosine

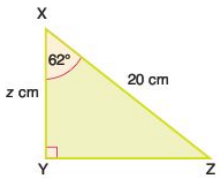
The cosine ratio is:

$$\cos Z = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$$



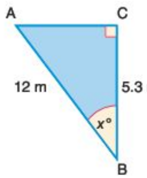
Worked examples

1 Calculate the length XY.



$$\cos 62° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{z}{20}$$
$$z = 20 \times \cos 62°$$
$$z = 9.39 \text{ (3 s.f.)}$$
$$XY = 9.39\,\text{cm (3 s.f.)}$$

2 Calculate the size of angle ABC.

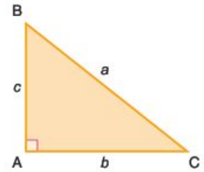


$$\cos x = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos x = \frac{5.3}{12}$$
$$x = \cos^{-1}\left(\frac{5.3}{12}\right)$$
$$x = 63.8 \text{ (3 s.f.)}$$
$$\angle ABC = 63.8° \text{ (3 s.f.)}$$

Pythagoras' theorem

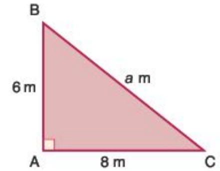
Pythagoras' theorem states the relationship between the lengths of the three sides of a right-angled triangle:

$$a^2 = b^2 + c^2$$



Worked examples

1 Calculate the length of the side BC.

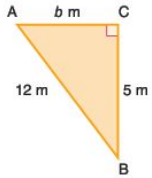


Using Pythagoras' theorem:

$$a^2 = b^2 + c^2$$
$$a^2 = 8^2 + 6^2$$
$$a^2 = 64 + 36 = 100$$
$$a = \sqrt{100}$$
$$a = 10$$

BC = 10 m

2 Calculate the length of the side AC.



Using Pythagoras' theorem:

$$a^2 = b^2 + c^2$$
$$a^2 - c^2 = b^2$$
$$b^2 = 144 - 25 = 119$$
$$b = \sqrt{119}$$
$$b = 10.9 \text{ (3 s.f.)}$$

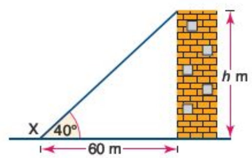
AC = 10.9 m (3 s.f.)

Angles of elevation and depression

The **angle of elevation** is the angle above the horizontal through which a line of view is raised. The **angle of depression** is the angle below the horizontal through which a line of view is lowered.

Worked examples

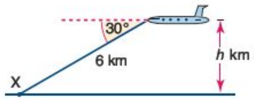
1 The base of a tower is 60 m away from a point X on the ground. If the angle of elevation of the top of the tower from X is 40° calculate the height of the tower. Give your answer to the nearest metre.



$$\tan 40° = \frac{h}{60}$$
$$h = 60 \times \tan 40° = 50$$

The height is 50 m.

2 An aeroplane receives a signal from a point X on the ground. If the angle of depression of point X from the aeroplane is 30°, calculate the height at which the plane is flying.
Give your answer to the nearest 0.1 kilometre.



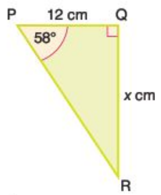
$\sin 30° = \frac{h}{6}$

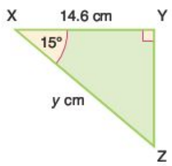
$h = 6 \times \sin 30° = 3.0$

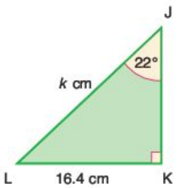
The height is 3.0 km.

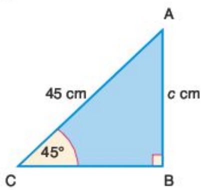
Exercise 5.3.1

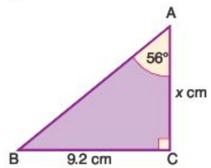
1 Calculate the unknown length in each of the diagrams.

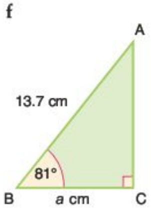
a


b


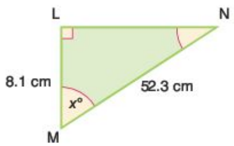
c


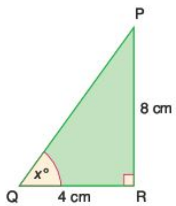
d


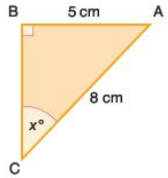
e


f


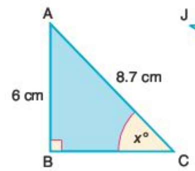
2 Calculate the size of the marked angle x° in each of the following diagrams.

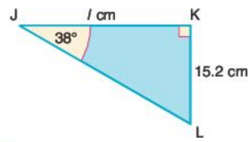
a


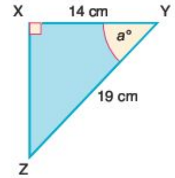
b


c


3 Calculate the unknown length or angle in each of the following diagrams

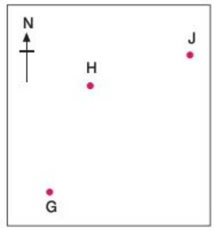
a


b


c


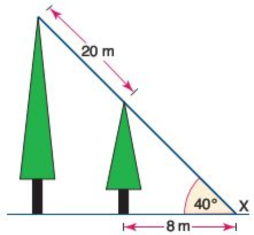
4 A sailing boat sets off from a point X and heads towards Y, a point 17 km north. At point Y it changes direction and heads towards point Z, a point 12 km away on a bearing of 090°. Once at Z the crew want to sail back to X. Calculate:
  a the distance ZX
  b the bearing of X from Z.

5 An aeroplane sets off from G on a bearing of 024° towards H, a point 250 km away. At H it changes course and heads towards J on a bearing of 055° and a distance of 180 km away.



  a How far is H to the north of G?
  b How far is H to the east of G?
  c How far is J to the north of H?
  d How far is J to the east of H?
  e What is the shortest distance between G and J?
  f What is the bearing of G from J?

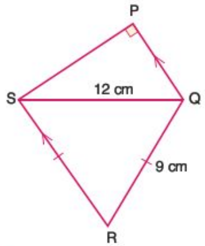
6 Two trees are standing on flat ground. The angle of elevation of their tops from a point X on the ground is 40°. If the horizontal distance between X and the small tree is 8 m and the distance between the tops of the two trees is 20 m, calculate:
  a the height of the small tree
  b the height of the tall tree
  c the horizontal distance between the trees.



7 PQRS is a quadrilateral. The sides RS and QR are the same length. The sides QP and RS are parallel.

Calculate:

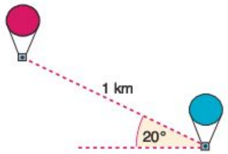
a angle SQR
b angle PSQ
c length PQ
d length PS
e the area of PQRS.



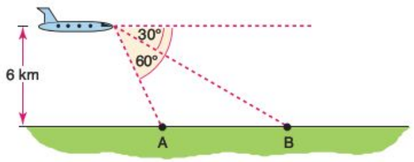
8 Two hot air balloons are 1 km apart in the air. If the angle of elevation of the higher from the lower balloon is 20°, calculate:

a the vertical height between the two balloons
b the horizontal distance between the two balloons.

Give your answers to the nearest metre.

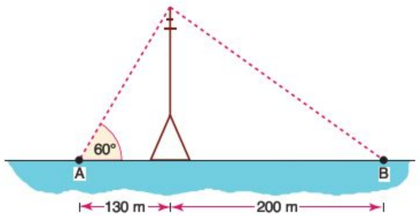


9 A plane is flying at an altitude of 6 km directly over the line AB. It spots two boats A and B on the sea. If the angles of depression of A and B from the plane are 60° and 30° respectively, calculate the horizontal distance between A and B.



10 Two people A and B are standing either side of a transmission mast. A is 130 m away from the mast and B is 200 m away. If the angle of elevation of the top of the mast from A is 60°, calculate:

a the height of the mast to the nearest metre
b the angle of elevation of the top of the mast from B.



5.4 Trigonometry and non-right-angled triangles

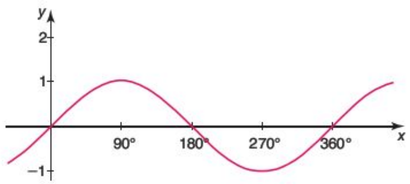
Angles between 0° and 180°

When calculating the size of angles using trigonometry, there is often more than one possible solution. Most calculators, however, will give only the first solution. To be able to calculate the value of a second possible solution, we need to look at the shape of trigonometrical graphs in more detail.

Note: The sine and cosine functions are also covered in Section 4.5.

The sine curve

The graph of $y = \sin x$ is plotted below for x in the range $0 \le x \le 360°$, where x is the size of the angle in degrees.

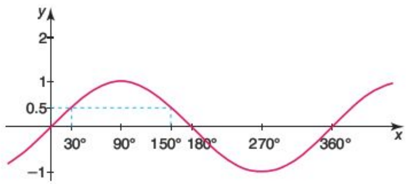


The graph of $y = \sin x$ has:

- a period of 360° (i.e. it repeats itself every 360°)
- a maximum value of 1 (at 90°)
- a minimum value of –1 (at 270°).

Worked example

$\sin 30° = 0.5$. Which other angle between 0° and 180° has a sine of 0.5?



From the graph above it can be seen that $\sin 150° = 0.5$.

$\sin x = \sin(180° - x)$

Exercise 5.4.1

1 Express each of the following in terms of the sine of another angle between 0° and 180°.

a $\sin 60°$	b $\sin 80°$	c $\sin 115°$
d $\sin 140°$	e $\sin 128°$	f $\sin 167°$

2 Express each of the following in terms of the sine of another angle between 0° and 180°.

a $\sin 35°$	b $\sin 50°$	c $\sin 30°$
d $\sin 48°$	e $\sin 104°$	f $\sin 127°$

3 Solve these equations for $0° \leq x \leq 180°$. Give your answers to the nearest degree.

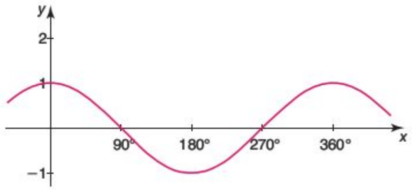
a $\sin x = 0.33$	b $\sin x = 0.99$	c $\sin x = 0.09$
d $\sin x = 0.95$	e $\sin x = 0.22$	f $\sin x = 0.47$

4 Solve these equations for $0° \leq x \leq 180°$. Give your answers to the nearest degree.

a $\sin x = 0.94$	b $\sin x = 0.16$	c $\sin x = 0.80$
d $\sin x = 0.56$	e $\sin x = 0.28$	f $\sin x = 0.33$

The cosine curve

The graph of $y = \cos x$ is plotted below for x in the range $0 \leq x \leq 360°$, where x is the size of the angle in degrees.



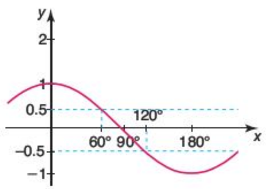
As with the sine curve, the graph of $y = \cos x$ has:

- a period of 360°
- a maximum value of 1 (at 0° and 360°)
- a minimum value of −1 (at 180°).

Note that $\cos x° = -\cos(180 - x)°$.

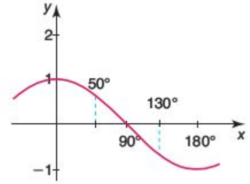
Worked examples

1 $\cos 60° = 0.5$. Which angle between 0° and 180° has a cosine of −0.5?



From the graph above it can be seen that $\cos 120° = -0.5$ as the curve is symmetrical.

2 The cosine of which angle between 0° and 180° is equal to the negative of $\cos 50°$?



$\cos 130° = -\cos 50°$

Exercise 5.4.2

1 Express each of the following in terms of the cosine of another angle between 0° and 180°.

a $\cos 20°$	b $\cos 85°$	c $\cos 32°$
d $\cos 95°$	e $\cos 147°$	f $\cos 106°$

2 Express each of the following in terms of the cosine of another angle between 0° and 180°.

a $\cos 98°$	b $\cos 144°$	c $\cos 160°$
d $\cos 143°$	e $\cos 171°$	f $\cos 123°$

3 Express each of the following in terms of the cosine of another angle between 0° and 180°.

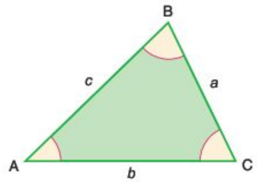
a $-\cos 100°$	b $\cos 90°$	c $-\cos 110°$
d $-\cos 45°$	e $-\cos 122°$	f $-\cos 25°$

4 The cosine of which acute angle has the same value as:

a $\cos 125°$	b $\cos 107°$	c $-\cos 120°$
d $-\cos 98°$	e $-\cos 92°$	f $-\cos 110°$?

The sine rule

With right-angled triangles we can use the basic trigonometric ratios of sine, cosine and tangent. The **sine rule** is a relationship which can be used with non-right-angled triangles.



The sine rule states that:

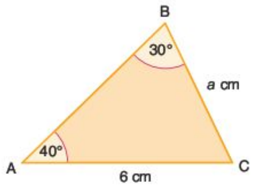
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

It can also be written as:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Worked examples

1 Calculate the length of side BC.



Using the sine rule:

$\frac{a}{\sin A} = \frac{b}{\sin B}$

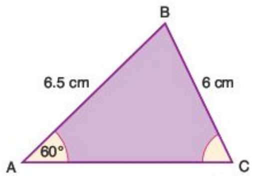
$\frac{a}{\sin 40°} = \frac{6}{\sin 30°}$

$a = \frac{6}{\sin 30°} \times \sin 40°$

$a = 7.71$ (3 s.f.)

BC = 7.71 cm (3 s.f.)

2 Calculate the size of angle C.



Using the sine rule:

$\frac{\sin A}{a} = \frac{\sin C}{c}$

$\frac{\sin 60°}{6} = \frac{\sin C}{6.5}$

$\sin C = \frac{6.5 \times \sin 60°}{6}$

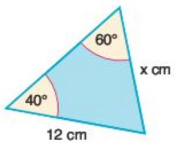
$C = \sin^{-1}(0.94)$

$C = 69.8°$ (3 s.f.)

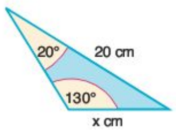
Exercise 5.4.3

1 Calculate the length of the side marked *x* in each of the following. Give your answers to one decimal place.

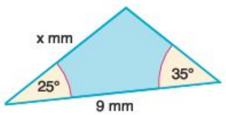
a



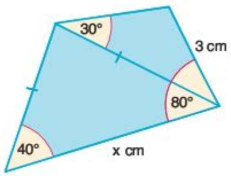
b



c

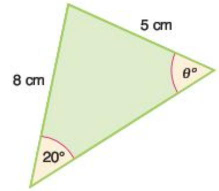


d

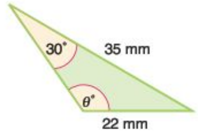


2 Calculate the size of the angle marked θ° in each of the following. Give your answers to one decimal place.

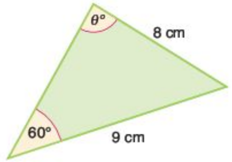
a



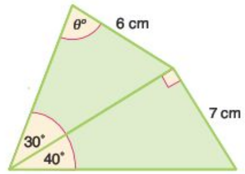
b



c



d



3 Triangle ABC has the following dimensions:

AC = 10 cm, AB = 8 cm and angle ACB = 20°.

a Calculate the two possible values for angle ABC.
b Sketch and label the two possible shapes for triangle ABC.

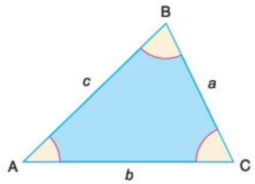
4 Triangle PQR has the following dimensions:

PQ = 6 cm, PR = 4 cm and angle PQR = 40°.

a Calculate the two possible values for angle QRP.
b Sketch and label the two possible shapes for triangle PQR.

The cosine rule

The **cosine rule** is another relationship which can be used with non-right-angled triangles.



The cosine rule states that:

$a^2 = b^2 + c^2 - 2bc \cos A$

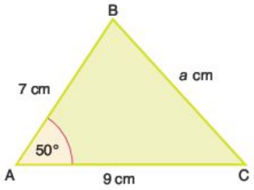
Worked examples

1 Calculate the length of the side BC.

Using the cosine rule:

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$a^2 = 9^2 + 7^2 - (2 \times 9 \times 7 \times \cos 50°)$$
$$= 81 + 49 - (126 \times \cos 50°) = 49.0$$
$$a = \sqrt{49.0}$$
$$a = 7.00 \text{ (3 s.f.)}$$

BC = 7.00 cm (3 s.f.)



2 Calculate the size of angle A.

Using the cosine rule:

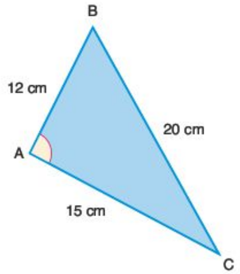
$$a^2 = b^2 + c^2 - 2bc \cos A$$

Rearranging the equation gives:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

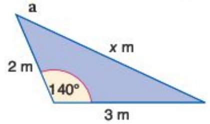
$$\cos A = \frac{15^2 + 12^2 - 20^2}{2 \times 15 \times 12} = 0.086$$

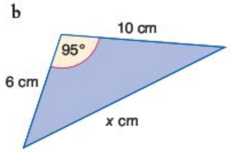
$$A = \cos^{-1}(-0.086)$$
$$A = 94.9° \text{ (3 s.f.)}$$

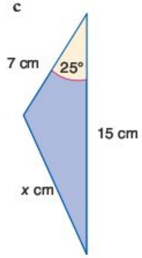


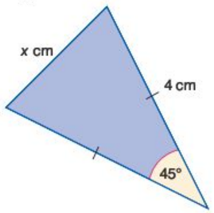
Exercise 5.4.4

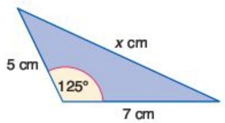
1 Calculate the length of the side marked x in each of the following. Give your answers to one decimal place.

a


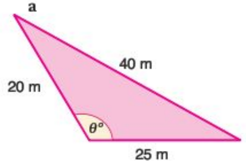
b


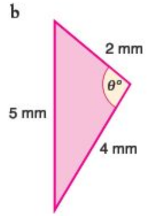
c


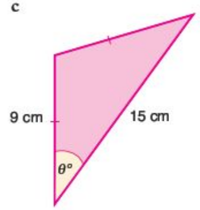
d


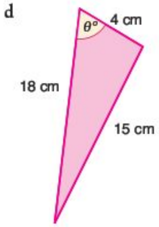
e


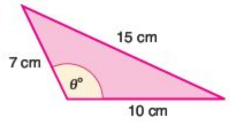
2 Calculate the angle marked $\theta°$ in each of the following. Give your answers to one decimal place.

a


b


c


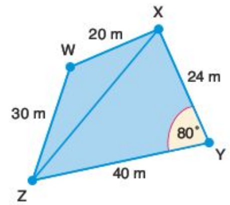
d


e


3 Four players W, X, Y and Z are on a rugby pitch. The diagram shows a plan view of their relative positions.

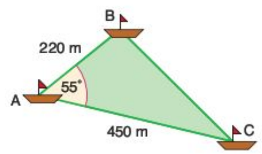
Calculate:

a the distance between players X and Z
b angle ZWX
c angle WZX
d angle YZX
e the distance between players W and Y.

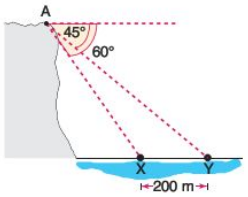


4 Three yachts A, B and C are racing off the 'Cape'. Their relative positions are shown in the diagram.

Calculate the distance between B and C to the nearest 10 m.



5 A girl standing on a cliff top at A can see two buoys X and Y, 200 m apart, floating on the sea. The angle of depression of Y from A is 45°, and the angle of depression of X from A is 60° (see diagram).

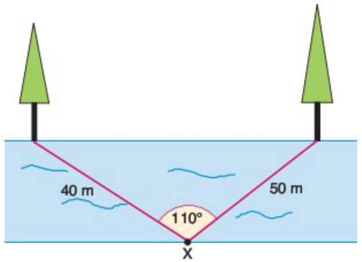


If A, X, Y are in the same vertical plane, calculate:

a the distance AY
b the distance AX
c the vertical height of the cliff.

6 There are two trees standing on one side of a river bank. On the opposite side is a boy standing at X.

Using the information given, calculate the distance between the two trees.

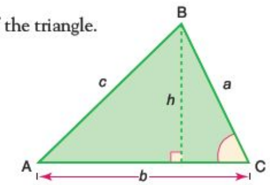


The area of a triangle

The area of a triangle is given by the formula:

Area $= \frac{1}{2}bh$

where b is the base and h is the vertical height of the triangle.



From trigonometric ratios we also know that:

$\sin C = \frac{h}{a}$

Rearranging, we have:

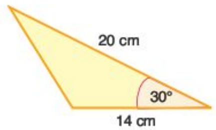
$h = a \sin C$

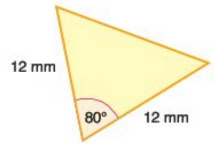
Substituting for h in the original formula gives another formula for the area of a triangle:

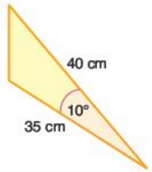
Area $= \frac{1}{2}ab \sin C$

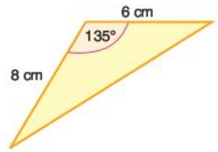
Exercise 5.4.5

1 Calculate the area of the following triangles. Give your answers to one decimal place.

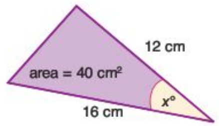
a


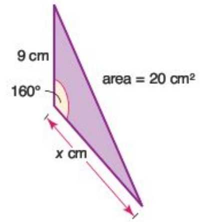
b


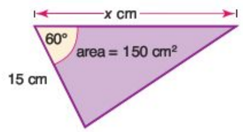
c


d


2 Calculate the value of x in each of the following. Give your answers correct to one decimal place.

a


b


c


d